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Scalar field probes of power-law space-time singularities

Matthias Blau, Denis Frank and Sebastian Weiss

Institut de Physique, Université de Neuchâtel
Rue Breguet 1, CH-2000 Neuchâtel, Switzerland
E-mail: matthias.blau@unine.ch, denis.frank@unine.ch, sebastian.weiss@unine.ch

Abstract: We analyse the effective potential of the scalar wave equation near generic space-time singularities of power-law type (Szekeres-Iyer metrics) and show that the effective potential exhibits a universal and scale invariant leading inverse square behaviour $\sim x^{-2}$ in the "tortoise coordinate" x provided that the metrics satisfy the strict Dominant Energy Condition (DEC). This result parallels that obtained in [1] for probes consisting of families of massless particles (null geodesic deviation, a.k.a. the Penrose Limit). The detailed properties of the scalar wave operator depend sensitively on the numerical coefficient of the x^{-2}-term, and as one application we show that timelike singularities satisfying the DEC are quantum mechanically singular in the sense of the Horowitz-Marolf (essential self-adjointness) criterion. We also comment on some related issues like the near-singularity behaviour of the scalar fields permitted by the Friedrichs extension.

Keywords: Penrose limit and pp-wave background, Black Holes.

Contents

1. Introduction

The study of scalar field propagation in non-trivial curved (and possibly singular) backgrounds is of fundamental importance in a variety of contexts including quantum field theory in curved backgrounds, cosmology, the stability and quasi-normal mode analysis of black hole metrics etc.

Typically, this is studied within the context of a particular metric or class of metrics. For certain purposes, however, only the knowledge of the leading behaviour of the metric near a horizon or the singularity is required. In that case, one can attempt to work with a general parametrisation of the metric near that locus and, in this way, ascertain which features of the results that have been obtained previously for particular metrics are special features of those metrics or valid more generally.

In particular, practically all explicitly known metrics with singularities are of "power-law type" [2] in a neighbourhood of the singularity (instead of showing, say, some non-analytic behaviour). In the spherically symmetric case, the leading behaviour of generic metrics with such singularities of power-law type is captured by the 2-parameter family

$$ds^2 = \eta x^p(-dx^2 + dy^2) + x^q d\Omega_d^2 \tag{1.1}$$

of Szekeres-Iyer metrics [2−4]. The singularity, located in these coordinates at $x = 0$, is timelike for $\eta = -1$ and spacelike for $\eta = +1$. This class of metrics thus provides an ideal laboratory for investigating the behaviour of particles, fields, strings, ... in the vicinity of a generic singularity of this type.

A first investigation along these lines was performed in [5, 1] in the context of the Penrose Limit, i.e. of probing a space-time via the geodesic deviation of families of massless particles. There it was shown that the plane wave Penrose limits,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \rightarrow 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \quad , \tag{1.2}$$

of metrics with singularities of power-law type have a universal u^{-2}-behaviour near the singularity, $A_{ab}(u) \sim u^{-2}$, provided that the near-singularity stress-energy (Einstein) tensor satisfies the strict dominant energy condition (DEC). This behaviour, which is precisely such that it renders the plane wave metric scale invariant [6], had previously been observed in various particular examples and is thus now understood to be a general feature of this class of singularities.

It is then natural to wonder whether a similar universality result can be established in other circumstances or for other kinds of probes and if, analogously, some energy condition plays a role in establishing this. If one considers e.g. the Klein-Gordon equation $\Box\phi = 0$ for scalar fields, it is not difficult to see [7, 8] that under certain conditions the scalar effective potential V_{eff} for general metrics with singularities of power-law type displays an inverse square behaviour, $V_{\text{eff}}(x) \sim x^{-2}$, near the singularity. This observation was then used in [8] to study the quasi-normal modes for black holes with generic singularities of this type.

The purpose of this note is to study other aspects and consequences of this universality. In particular, we will first show that the results obtained in [1], namely the scale invariant inverse square behaviour of the wave profile $A_{ab}(u)$, as well as a crucial [9, 6] lower bound on the coefficients, have a precise and rather striking analogue in the case of a scalar field. Schematically, this analogy can be expressed as

$$\text{strict DEC} \quad \Rightarrow \quad \begin{cases} A_{ab}(u) \rightarrow c_a \delta_{ab} u^{-2} & \text{scale invariant } (c_a \geq -1/4) \\ V_{\text{eff}}(x) \rightarrow c x^{-2} & \text{scale invariant } (c \geq -1/4) \end{cases} \tag{1.3}$$

Once again this shows that this inverse square behaviour, that had been observed before in various specific examples in a variety of contexts, is a general feature of a large class of space-time singularities. The precise statements are derived in sections 2.2 and 2.3 and discussed in section 2.4, while sections 2.5 and 2.6 deal with minor variations of this theme.

We hasten to add that if such an inverse square behaviour were universally true without any further qualifications then it would probably have to be true on rather trivial (dimensional) grounds alone. What makes the results obtained here and in [1] more interesting is that a priori in either case a more singular behaviour can and does occur and is only excluded provided that some further (e.g. positive energy) condition is imposed.

The significance of the x^{-2}-behaviour is that (as anticipated in (1.3)), the corresponding Schrödinger operator $-\partial_x^2 + cx^{-2}$, to which we will have reduced the Klein-Gordon operator, defines a scale invariant (c is dimensionless) "conformal quantum mechanics" [10] problem. Thus, here and in [1] we find a rather surprising emergence of scale invariance in the near-singularity limit. One implication of this scale invariance in the plane wave

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case, discussed in [11], is that it leads to a Hagedorn-like behaviour of string theory in this class of backgrounds that is quite distinct from that in plane wave backgrounds with, say, a constant profile and more akin to that in Minkowski space. It would be interesting to explore other consequences of this near-singularity scale invariance.

This class of scale invariant models has recently also appeared and been discussed in various other related settings, most notably in the analysis of the near-horizon (rather than the near-singularity) properties of black holes, see e.g. [12−16], where the emergence of scale invariance can largely be attributed to the near-horizon AdS geometry, as well as in quantum cosmology [17].

Having reduced the Klein-Gordon operator to the Schrödinger operator $-\partial_x^2 + cx^{-2}$ (after a separation of variables and a unitary transformation), one can then turn to a more detailed spectroscopy of the Szekeres-Iyer metrics by analysing the properties of this operator. Indeed, as is well known, the inverse square potential is a critical borderline case in the sense that the functional analytic properties of this operator depend in a delicate way on the numerical value of the coefficient c. This value, in turn, depends on the dimension d (number of transverse dimensions) and the Szekeres-Iyer parameter q (it turns out to be independent of p, while the corresponding coefficients c_a in the Penrose limit case typically depend on (p, q) and d).

As one application, we will analyse the Horowitz-Marolf criterion [18] for general singularities of power-law type. Horowitz and Marolf defined a static space-time to be quantum mechanically non-singular (with respect to a certain class of test fields) if the evolution of a probe wave packet is uniquely determined by the initial wave packet (as would be the case in a globally hyperbolic space-time) without having to specify boundary conditions at the classical singularity. This criterion can be rephrased as the condition that the (spatial part of the) Klein-Gordon operator be essentially self-adjoint (and thus have a unique self-adjoint extension).

While such a necessarily only semi-classical analysis is certainly not a substitute for a full quantum gravitational analysis, it nevertheless has its virtues since one can learn what kind of problems persist, can arise or can be resolved when passing from test particles to test fields.

Intuitively one might expect a classical singularity with a sufficiently "positive" (in an appropriate sense) matter content to remain singular even when probed by non-stringy test objects other than classical point particles. This line of thought was one of the motivations for analysing the Horowitz-Marolf criterion in this framework, and we will indeed be able to show (section 3.4) that

> metrics with timelike singularities of power-law type satisfying the strict Dominant Energy Condition remain singular when probed with scalar waves.

A second issue we will briefly address is that of the allowed near-singularity behaviour of the scalar fields for a given self-adjoint extension (section 3.5). A priori, one might perhaps expect a sufficiently repulsive singularity to be regular in the Horowitz-Marolf sense simply because the corresponding unique self-adjoint extension forces the scalar field to be zero

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at the singularity, thus in a sense again excluding the singularity from the space-time. It is also possible, however, and potentially more interesting, to have a self-adjoint extension with scalar fields that actually probe the singularity in the sense that they are allowed to take on non-zero values there. We propose to call such singularities "hospitable", establish once again a relation, albeit not a strict correlation, with the DEC, and show among other things that, in a suitable sense, half of the Horowitz-Marolf regular power-law singularities are hospitable whereas the others are not.

2. Universality of the effective scalar potential for power-law singularities

2.1 Geometric set-up

Even though we will ultimately be interested in the properties of the scalar wave (Klein-Gordon) equation $(\Box - m^2)\phi = 0$ in the Szekeres-Iyer metrics (1.1), to set the stage it will be convenient to begin the discussion in the more general setting of metrics with a hypersurface orthogonal Killing vector. The general set-up here and in section 3.1 is modelled on the approach of [19] (with minor adaptations to allow for both timelike and spacelike singularities).

We begin with the n-dimensional metric

$$ds^2 = \eta a^2 dy^2 + h_{ij} dx^i dx^j \tag{2.1}$$

where a and h_{ij} are independent of y, $\xi = \partial_y$ is a hypersurface orthogonal Killing vector with norm $\xi^\mu \xi_\mu = \eta a^2$, and thus timelike (spacelike) for $\eta = -1$ ($\eta = +1$). Correspondingly we assume that the metric h_{ij} induced on the hypersurfaces $\Sigma_y \cong \Sigma$ of constant y is Riemannian (Lorentzian) for $\eta = -1$ ($\eta = +1$).

Denoting the covariant derivatives with respect to the metric h_{ij} by D_i, the wave operator

$$\Box \equiv \frac{1}{\sqrt{-\det g}} \partial_\mu \sqrt{-\det g}\, g^{\mu\nu} \partial_\nu \tag{2.2}$$

is easily seen to take the form

$$\Box = a^{-2}(\eta \partial_y^2 + a D^i a D_i) \ . \tag{2.3}$$

Thus the massive wave equation $(\Box - m^2)\phi = 0$ can be written as

$$\partial_y^2 \phi = -A\phi \ , \tag{2.4}$$

where A is the operator

$$A = \eta a D^i a D_i - \eta a^2 m^2 \ . \tag{2.5}$$

Assuming now spherical symmetry, the metric takes the warped product form

$$ds^2 = \eta a(x)^2 dy^2 - \eta b(x)^2 dx^2 + c(x)^2 d\Omega_d^2 \tag{2.6}$$

where $d\Omega_d^2$, $d = n - 2$, denotes the standard metric on the d-sphere S^d. It will be apparent from the following that the assumption of spherical symmetry could be relaxed - we will only use the warped product form of the metric in an essential way.

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We could fix the residual x-reparametrisation invariance by introducing the "area radius" $r = c(x)$ as a new coordinate. However, for the following it will be more convenient to choose the gauge $a(x) = b(x)$ (i.e. x is a "tortoise coordinate" for $\eta = -1$ respectively "conformal time" for $\eta = +1$),

$$ds^2 = \eta a(x)^2(-dx^2 + dy^2) + c(x)^2 d\Omega_d^2 \ . \tag{2.7}$$

Then the operator A is

$$A = -\sigma^{-1}\partial_x \sigma \partial_x + \eta a^2 c^{-2}\Delta_d - \eta a^2 m^2 \ , \tag{2.8}$$

where $\sigma(x) = c(x)^d$ and Δ_d denotes the Laplacian on S^d.

To put A into standard Schrödinger form, we transform from the functions $\phi(x)$ to the half-densities (cf. (3.8)) $\tilde{\phi}(x) = \sigma^{1/2}\phi(x)$. The corresponding unitarily transformed operator $\tilde{A}$ is

$$\begin{aligned} \tilde{A} &= \sigma^{1/2} A \sigma^{-1/2} = -\partial_x^2 + V + \eta a^2 c^{-2}\Delta_d - \eta a^2 m^2 \\ V(x) &= \sigma(x)^{-1/2}(\partial_x^2 \sigma(x)^{1/2}) \ . \end{aligned} \tag{2.9}$$

After the usual separation of variables in the y-direction,

$$\tilde{\phi}(y, x, \theta^a) = \mathrm{e}^{-iEy}\ \tilde{\phi}(x, \theta^a) \ , \tag{2.10}$$

and the decomposition into angular spherical harmonics $Y_{\ell\vec{m}}(\theta^a)$, with

$$\begin{aligned} -\Delta_d Y_{\ell\vec{m}}(\theta^a) &= \ell_d^2 Y_{\ell\vec{m}}(\theta^a) \\ \ell_d^2 &= \ell(\ell + d - 1) \ , \end{aligned} \tag{2.11}$$

the Klein-Gordon equation for the metric (2.7) reduces to a standard one-dimensional time-independent Schrödinger equation

$$\left[-\partial_x^2 + V_{\text{eff},\ell}(x)\right]\tilde{\phi}(x) = E^2\tilde{\phi}(x) \tag{2.12}$$

($\tilde{\phi}(x) = \tilde{\phi}_{E,\ell,\vec{m}}(x)$) with effective scalar potential

$$V_{\text{eff},\ell}(x) = V(x) - \eta a(x)^2(\ell_d^2 c(x)^{-2} + m^2) \ . \tag{2.13}$$

2.2 The effective scalar potential for power-law singularities

The leading behaviour of generic (spherically symmetric) metrics with singularities of power-law type,[1] i.e. metrics of the general form [2]

$$ds^2 = -dt^2 + [t - \tau(r)]^{2a} f(r,t)^2 dr^2 + [t - \tau(r)]^{2b} g(r,t)^2 d\Omega_d^2 \ , \tag{2.14}$$

[1] Such metrics encompass practically all explicitly known singular spherically symmetric solutions of the Einstein equations like the Lemaître-Tolman-Bondi dust solutions, cosmological singularities of the Lifshitz-Khalatnikov type, etc. On the other hand, this class of metrics does prominently *not* include the BKL metrics [20] describing the chaotic oscillatory approach to a spacelike singularity. Whether or not such a behaviour occurs depends in a delicate way on the matter content, see e.g. [21] and references therein.

with f and g functions of r and t that are regular and non-vanishing at the location $t = \tau(r)$ of the singularity, is captured by the 2-parameter family of Szekeres-Iyer metrics [2, 3] (see also [1] and the generalisation to string theory backgrounds discussed in [4])

$$ds^2 = \eta x^p(-dx^2 + dy^2) + x^q d\Omega_d^2 \ . \tag{2.15}$$

The Kasner-like exponents $p, q \in \mathbb{R}$ characterise the behaviour of the geometry near the singularity at $x = 0$. This singularity is timelike for $\eta = -1$ (x is a radial coordinate) and spacelike for $\eta = +1$ (with x a time coordinate). In particular, these metrics possess the hypersurface orthogonal Killing vector ∂_y, and are already in the "tortoise" form (2.7), with $a(x)^2 = x^p$ and $c(x)^2 = x^q$. Thus we can directly read off the effective scalar potential from the results of the previous section.

From (2.9), we deduce, with $\sigma(x) = x^{dq/2}$, that

$$V(x) = s(s-1)x^{-2} \qquad s = \frac{dq}{4} \ . \tag{2.16}$$

Thus, from (2.13) we find (see also [8])

$$V_{\text{eff},\ell}(x) = s(s-1)x^{-2} - \eta\ell_d^2 x^{p-q} - \eta m^2 x^p \tag{2.17}$$

We are interested in the leading behaviour of this potential as $x \to 0$ (subdominant terms can in any case not be trusted as we have only kept the leading terms in the metric (2.15)). For the time being we will consider the massless case $m^2 = 0$ (see section 2.5 for $m^2 \neq 0$).

Provided that $s(s-1) \neq 0$, which term in (2.17) dominates depends on p and q. When $q < p+2$, one finds

$$q < p+2: \qquad V_{\text{eff},\ell}(x) \to s(s-1)x^{-2} \ . \tag{2.18}$$

The two salient features of this potential are the inverse square behaviour and a coefficient c that is bounded from below by $-1/4$,

$$c = s(s-1) \geq -\frac{1}{4} \ , \tag{2.19}$$

with equality for $s = 1/2$, i.e. $q = 2/d$.

As mentioned in the introduction, the significance of the x^{-2}-behaviour is that it defines a scale invariant "conformal quantum mechanics" [10] problem, discussed more recently in related contexts e.g. in [12–17]. Moreover, for practical purposes [8, 22] the virtue of the x^{-2} (as opposed to a more singular) behaviour is that it leads to a standard regular-singular differential operator.

The significance of the bound on c is that in this range the operator $-\partial_x^2 + c/x^2$ is positive, as can be seen by writing

$$-\partial_x^2 + s(s-1)x^{-2} = (\partial_x + sx^{-1})(-\partial_x + sx^{-1}) = (-\partial_x + sx^{-1})^\dagger(-\partial_x + sx^{-1}) \ . \tag{2.20}$$

When $q = p+2$, the metric is conformally flat, both terms in (2.17) contribute equally, and one again finds the x^{-2}-behaviour

$$q = p+2: \qquad V_{\text{eff},\ell}(x) \to cx^{-2} \ , \tag{2.21}$$

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where now

$$c = s(s-1) - \eta\ell_d^2 \ . \tag{2.22}$$

Thus in this case c is still bounded by $-1/4$ for timelike singularities, while c can become arbitrarily negative for sufficiently large values of ℓ_d^2 for $\eta = +1$.

Once $q > p+2$, the second term in (2.17) dominates (for $\ell_d^2 \neq 0$), and one finds the more singular leading behaviour

$$q > p+2: \qquad V_{\text{eff},\ell}(x) \to -\eta\ell_d^2\, x^{-2-a} \quad a > 0 \ . \tag{2.23}$$

Examples of metrics with $q \leq p+2$ are the Schwarzschild and Friedmann-Robertson-Walker (FRW) metrics and indeed, as we will recall below, all metrics satisfying the strict Dominant Energy Condition.

In particular, for the $(d+2)$-dimensional (positive or negative mass) Schwarzschild metric, one has

$$\text{Schwarzschild}: \qquad p = \frac{1-d}{d} \qquad q = \frac{2}{d} \ , \tag{2.24}$$

as is readily seen by expanding the metric near the singularity and going to tortoise coordinates. Thus the Schwarzschild metric has $s = 1/2$ and c takes on the d-independent extremal value $c = -1/4$, as observed before e.g. in [22, 8] in related contexts.

For decelerating cosmological FRW metrics, with cosmological scale factor (in comoving time) $\sim t^h$, $0 < h < 1$,

$$h = \frac{2}{(d+1)(1+w)} \ , \tag{2.25}$$

with w the equation of state parameter, $P = w\rho$, one finds [5, 1]

$$\text{FRW}: \qquad p = q = \frac{2h}{1-h} \ , \tag{2.26}$$

as can be seen by going to conformal time. A routine calculation shows that the above result (2.18) for the purely x-dependent part of the effective potential (with x conformal time) is actually an exact result, and not an artefact of the near-singularity Szekeres-Iyer approximation.

It remains to discuss the case when $q < p+2$, so that the first term in (2.17) would be dominant, but the coefficient $s(s-1) = 0$. When $s = 0$, then one has $q = 0$ and this is generally interpreted [2] as corresponding not to a true central singularity (as the radius of the transverse sphere remains constant as $x \to 0$) but as a shell crossing singularity.

The other possibility is $s = 1$, i.e. $q = 4/d$. This is a case in which (because of the cancellation of the leading terms) subleading corrections to the metric (2.15) can become relevant and should be retained. An example of metrics with $s = 1$ is provided by FRW metrics with a radiative equation of state. Using (2.26), one has

$$q = \frac{4}{d} \Leftrightarrow h = \frac{2}{d+2} \Leftrightarrow w = \frac{1}{d+1} \ , \tag{2.27}$$

which is precisely the equation of state parameter for radiation. However, as follows from the remark above, in this special case the vanishing of the effective potential for $p = q$ is actually an exact result.

2.3 The significance of the (strict) dominant energy condition

We have seen that generically the leading behaviour of the scalar effective potential near a singularity of power-law type is either $\sim x^{-2}$ or $\sim x^{p-q}$. We will now recall from [2, 1] that the latter behaviour can arise only for metrics violating the strict Dominant Energy Condition (DEC). While there is nothing particularly sacrosanct about the DEC, and other energy conditions could be considered, the DEC appears to play a privileged role in exploring and understanding the (p,q)-plane of Szekeres-Iyer metrics.

The *Dominant Energy Condition* on the stress-energy tensor $T^{\mu}_{\ \nu}$ (or Einstein tensor $G^{\mu}_{\ \nu}$) [23] requires that for every timelike vector v^{μ}, $T_{\mu\nu}v^{\mu}v^{\nu} \geq 0$, and $T^{\mu}_{\ \nu}v^{\nu}$ be a non-spacelike vector. This may be interpreted as saying that for any observer the local energy density is non-negative and the energy flux causal.

The Einstein tensor of Szekeres-Iyer metrics is diagonal, hence so is the corresponding stress-energy tensor. In this case, the DEC reduces to

$$\rho \geq |P_i| \ , \tag{2.28}$$

where $-\rho$ and P_i, $i = 1, \ldots, d+1$ are the timelike and spacelike eigenvalues of $T^{\mu}_{\ \nu}$ respectively. We say that the *strict* DEC is satisfied if these are strict inequalities and we will say that the matter content (or equation of state) is "extremal" if at least one of the inequalities is saturated.

Now it follows from the explicit expression for the components

$$\begin{aligned} G^x_x &= -\frac{1}{2}d(d-1)x^{-q} - \frac{1}{8}\eta dq((d-1)q + 2p)x^{-(p+2)} \\ G^y_y &= -\frac{1}{2}d(d-1)x^{-q} + \frac{1}{8}\eta dq(2p + 4 - (d+1)q)x^{-(p+2)} \end{aligned} \tag{2.29}$$

of the Einstein tensor that for $q > p+2$ the relation between $-\rho$ and the radial pressure P_r (identified with G^x_x and G^y_y - which is which depends on the sign of η) becomes extremal as $x \to 0$ [2, 1],

$$q > p+2: \qquad G^x_x - G^y_y \to 0 \quad \Leftrightarrow \quad \rho + P_r \to 0 \ . \tag{2.30}$$

Put differently, $q \leq p+2$ is a necessary condition for the strict DEC to hold, and thus for metrics satisfying the strict DEC the leading behaviour of the effective potential is always $V_{\text{eff},\ell}(x) \to cx^{-2}$.

As an aside, we note that it follows from (2.29) that precisely those metrics that satisfy the physically more reasonable (non-negative pressure) and more common extremal near-singularity equation of state $\rho = +P_r$ have $q = 2/d$, i.e. $s = 1/2$, leading to the critical value $c = -1/4$ frequently found in applications (to e.g. Schwarzschild-like geometries).

2.4 Comparison with massless point particle probes (the Penrose limit)

In the previous section we have established that

1. for metrics with singularities of power-law type satisfying the strict DEC the leading behaviour of the scalar effective potential near the singularity is

$$V_{\text{eff},\ell}(x) \to cx^{-2} \tag{2.31}$$

2. this class of potentials is singled out by its scale invariance;

3. the corresponding coefficient c of the effective potential is bounded from below by $-1/4$ unless one is on the border to an extremal equation of state.

These observations bear a striking resemblance to the results obtained recently in [1] in the study of plane wave Penrose limits

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \to 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \ , \tag{2.32}$$

of space-time singularities. Namely, it was shown in [1] that

1. Penrose limits of metrics with singularities of power-law type show a universal u^{-2}-behaviour near the singularity,

$$A_{ab}(u) \to c_a \delta_{ab} u^{-2} \ , \tag{2.33}$$

provided that the strict DEC is satisfied;

2. such plane waves are singled out [6] by their scale invariance, reflected e.g. in the isometry $(u,v) \to (\lambda u, \lambda^{-1}v)$ of the metric (2.32), (2.33);

3. the coefficients c_a (related to the harmonic oscillator frequency squares by $c_a = -\omega_a^2$) are bounded from below by $-1/4$ unless one is on the border to an extremal equation of state.[2]

The similarity of these two sets of statements is quite remarkable because the objects these statements are made about are rather different. For example, the potential is that of a one-dimensional motion on the half line in one case, and that of a d-dimensional harmonic oscillator (with time-dependent frequencies) in the other.

The analogy with the above statements about scalar effective potentials is brought out even more clearly if one reinterprets [5, 1] the Penrose limit in terms of null geodesic deviation in the original space-time. Then this result can be rephrased as the statement that the leading behaviour of the geometry as probed by a family of massless point particles near a singularity is that of a plane wave with a u^{-2} geodesic effective potential. The analogy with the results of the previous section should now be apparent.

One minor difference between the results obtained here and those of [1] is that in the case of Penrose limits the strict DEC needed to be invoked only in the case of spacelike singularities, $\eta = +1$, timelike singularities always giving rise to plane waves with a u^{-2}-behaviour. This should be regarded as an indication (cf. the discussion in [1, section 4.4]) that scalar waves are better probes of timelike singularities than massless point particles.

[2] One significance of this bound on the c_a is that in this range one can consider the possibility to extend the string modes across the singularity at $u = 0$ [9].

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2.5 Massive scalar fields and geodesic incompleteness

The simple above analysis can evidently be generalised in various ways, e.g. by considering other kinds of probes. We will briefly comment on the two most immediate generalisations, namely massive and non-minimally coupled scalar fields.

We begin with a massive scalar for which the effective potential is

$$V_{\text{eff},\ell}(x) = s(s-1)x^{-2} - \eta\ell_d^2 x^{p-q} - \eta m^2 x^p \tag{2.34}$$

For the mass term to be relevant (dominant) as $x \to 0$ it is clearly necessary that $p < -2$ and $q < 0$. Intuitively one might expect a mass term to be irrelevant at short distances near a singularity. This expectation is indeed borne out: as we will now show, for metrics satisfying the above inequalities the would-be singularity at $x = 0$ is at infinite affine distance for causal geodesics so that such space-times are actually causally geodesically complete.

Null geodesics were analysed in [1]. Here we generalise this to causal geodesics. In terms of the conserved angular and y-momentum L and P, the geodesic equation for the metric (2.15) reduces to

$$\dot{x}^2 = P^2 x^{-2p} + \eta L^2 x^{-p-q} + \eta\epsilon x^{-p} \ , \tag{2.35}$$

where $\epsilon = 0$ ($\epsilon = 1$) for null (timelike) geodesics respectively.

For $\eta = -1$, if the first term in (2.35) is sub-dominant the geodesic effective potential is repulsive (e.g. via the angular momentum barrier) and the geodesics will not reach $x = 0$. Thus generic timelike geodesics will reach $x = 0$ only if (p, q) lie in the positive wedge bounded by the lines $p = 0$ and $p = q$. Radial null geodesics do not feel any repulsive force, and solving

$$\dot{x}^2 \sim x^{-2p} \quad \Rightarrow \quad x(u) \sim \begin{cases} u^{1/(p+1)} & p \neq -1 \\ \exp u & p = -1 \end{cases} \tag{2.36}$$

shows that $x = 0$ is reached at a finite value of the affine parameter only for $p > -1$. We thus conclude that Szekeres-Iyer metrics with $\eta = -1$ and $p \leq -1$ are causally geodesically complete. In particular, therefore, the mass term in the scalar effective potential is sub-dominant for metrics with honest timelike power-law singularities, and for all such metrics the scalar effective potential has the same leading behaviour as in the massless case.

For $\eta = +1$, the situation is more complex as all three terms in (2.35) are positive. If the first term dominates, either because of suitable inequalities satisfied by (p, q) or, for any (p, q), because one is considering radial null geodesics, the analysis and conclusions are identical to the above. Namely, $x = 0$ is at finite affine distance for $p > -1$. Analogously, if the second term dominates (e.g. for angular null geodesics) one finds the condition $p + q > -2$, and if the third term dominates one has $p > -2$. Since one needs $p < -2$ for the mass term to dominate in the scalar effective potential, only the second case is possible. But then the condition $p + q > -2$, with $p < -2$, implies $q > 0$, so that the angular momentum term in the effective potential dominates the mass term.

We thus conclude that, for both $\eta = +1$ and $\eta = -1$, the mass term is always sub-dominant for metrics that are causally geodesically incomplete at $x = 0$.

As an aside we note that the Szekeres-Iyer metrics for which the mass term does dominate ($p < -2$ and $q < 0$), in addition to being non-singular, also necessarily violate the strict DEC.

2.6 Non-minimally coupled scalar fields

We will now very briefly also consider a non-minimally coupled scalar field

$$(\Box - \xi R)\phi = 0 \ . \tag{2.37}$$

The Ricci scalar of the Szekeres-Iyer metric (2.15) is

$$R = d(d-1)x^{-q} - \frac{1}{4}\eta(4p + 4qd - d(d+1)q^2)x^{-(p+2)} \ , \tag{2.38}$$

where once again only the leading order term should be trusted and retained. Thus the new effective potential

$$V^{\xi}_{\text{eff},\ell}(x) = V_{\text{eff},\ell}(x) - \eta\xi x^p R \tag{2.39}$$

is again a sum of two terms, proportional to x^{-2} and x^{p-q} respectively, so that the dominant behaviour is still $\sim x^{-2}$ provided that the metric does not violate the strict DEC. For $q < p+2$ and the conformally invariant coupling

$$\xi = \xi_* = \frac{d}{4(d+1)} \ , \tag{2.40}$$

one finds

$$V^{\xi_*}_{\text{eff},\ell}(x) = \frac{(p-q)d}{4(d+1)}x^{-2} = (p-q)\xi_* x^{-2} \ . \tag{2.41}$$

Note that with this conformally invariant coupling the coefficient c now depends on $p - q$ rather than on q. The appearance of $(p-q)$ could have been anticipated since for $p = q$ the Szekeres-Iyer metric is conformal to an x-independent metric, and hence a conformal coupling cannot generate an x-dependent effective potential. Note also that for the conformal coupling (and, indeed, generic values of ξ) the coefficient c is no longer bounded by $-1/4$ so that the Schrödinger operator is no longer necessarily bounded from below.

3. Self-adjoint physics of power-law singularities

In the previous section we have determined the leading behaviour of the scalar wave operators near a power-law singularity. In this section we will now study various aspects of these operators.

3.1 Functional analysis set-up

In order to analyse the properties of the wave operator, we will need to equip the space of scalar fields with a Hilbert space structure. We will be pragmatic about this and introduce the minimum amount of structure necessary to be able to say anything of substance.

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We thus return to the discussion of section 2.1, now being more specific about the spaces of functions the various operators appearing there act on [19], beginning with the operator A introduced in (2.5),

$$A = \eta a D^i a D_i - \eta a^2 m^2 \ . \tag{3.1}$$

Since $D^i D_i$ is symmetric (formally self-adjoint) with respect to the natural spatial density $\sqrt{-\eta \det h}$ induced on the slices Σ of constant y by the metric (2.1), the operator A is symmetric with respect to the scalar product

$$\begin{aligned}(\phi_1, \phi_2) &= \int d^{n-1}x \; \sigma \phi_1^* \phi_2 \\ \sigma &= a^{-1}\sqrt{-\eta \det h} = \eta \sqrt{-\det g}\, g^{yy} \ ,\end{aligned} \tag{3.2}$$

on $D(A) = C_0^\infty(\Sigma)$,

$$(A\phi_1, \phi_2) = (\phi_1, A\phi_2) \ . \tag{3.3}$$

Moreover, for $\eta = -1$ the operator A is positive,

$$\eta = -1 \quad \Rightarrow \quad (\phi, A\phi) \geq 0 \ . \tag{3.4}$$

We are thus led to introduce the Hilbert space $L^2(\Sigma, \sigma d^{n-1}x)$ of functions on Σ square integrable with respect to the above scalar product.

Passing to spherically symmetric metrics (2.6) in the tortoise gauge (2.7), A takes the form (2.8)

$$A = -\sigma^{-1}\partial_x \sigma \partial_x + \eta a^2 c^{-2} \Delta_d - \eta a^2 m^2 \ , \tag{3.5}$$

where $\sigma(x) = c(x)^d$. Since A is symmetric with respect to the scalar product (3.2), the unitarily transformed operator

$$\tilde{A} = \sigma^{1/2} A \sigma^{-1/2} \ , \tag{3.6}$$

acting on the half-densities

$$\tilde{\phi}(x) = \sigma(x)^{1/2}\phi(x) \ , \tag{3.7}$$

is symmetric with respect to the corresponding "flat" ($\sigma(x) \to 1$) scalar product

$$\langle \tilde{\phi}_1, \tilde{\phi}_2 \rangle := \int dx\, d\Omega \; \tilde{\phi}_1^* \tilde{\phi}_2 = (\phi_1, \phi_2) \ . \tag{3.8}$$

We now assume that the metric develops a singularity at some value of x, where e.g. the area radius goes to zero, $r \equiv c(x) \to 0$, which we may as well choose to happen at $x = 0$. Thus we consider $x \in (0, \infty)$ and take $\Sigma = \mathbb{R}^{n-1} \backslash \{0\}$, parametrised by x and the angular coordinates.

Then the initial domain of $\tilde{A}$ is $D(\tilde{A}) = C_0^\infty(\mathbb{R}^{n-1}\backslash\{0\})$ or $\tilde{D}(\tilde{A}) = C_0^\infty(\mathbb{R}_+) \otimes C^\infty(S^d)$, which are dense in the unitarily transformed Hilbert space

$$L^2(\mathbb{R}^{n-1}\backslash\{0\}, dx\, d\Omega) \cong L^2(\mathbb{R}_+, dx) \otimes L^2(S^d, d\Omega) \ . \tag{3.9}$$

Decomposing the second factor into eigenspaces of the Laplacian Δ_d on S^d,

$$L^2(\mathbb{R}_+, dx) \otimes L^2(S^d, d\Omega) = \bigoplus_{\ell=0}^{\infty} L_\ell \ , \tag{3.10}$$

and defining $\tilde{D}_\ell = \tilde{D} \cap L_\ell$, one has

$$\tilde{A}|_{\tilde{D}_\ell} = \tilde{A}_\ell \otimes \mathbb{I} \ , \tag{3.11}$$

where

$$\tilde{A}_\ell = -\partial_x^2 + V_{\text{eff},\ell}(x) \tag{3.12}$$

with $V_{\text{eff},\ell}(x)$ given in (2.13).

Questions about the original operator A can thus be reduced to questions about the family $\{\tilde{A}_\ell\}$ of standard Schrödinger-type operators. For example, to show that A is essentially self-adjoint on $D(A)$ it is sufficient to prove that, for each ℓ, $\tilde{A}_\ell$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+)$.

While one can analyse this question of self-adjointness just as readily for $\eta = +1$ as for $\eta = -1$, the physical significance of this condition in the case of spacelike singularities is not clear to us. Thus we will focus on static space-times with timelike singularities in the following and set $\eta = -1$. An extension of the general formalism to stationary non-static space-times is developed in [24].

We conclude this section with a comment on the choice of Hilbert space structure. The L^2 Hilbert space introduced above is certainly a natural choice, but not the only one possible. Based on physical requirements such as the finiteness of the energy of scalar field probes, other (Sobolev) Hilbert space structures have been proposed in the literature - see e.g. [25, 26]. The energy is, by definition,

$$E[\phi] = \int_\Sigma \sqrt{h} d^{n-1}x \ T_{\mu\nu}(\phi)\xi^\mu n^\nu \ , \tag{3.13}$$

where $T_{\mu\nu}(\phi)$ is the stress energy tensor of the scalar field, $\xi = \partial_y$ is the timelike Killing vector, and n the unit normal to Σ. In the present case this reduces to

$$E[\phi] = \int_\Sigma \sigma d^{n-1}x \ T_{yy} \ , \tag{3.14}$$

which identifies T_{yy} as the energy density with respect to the measure $\sigma d^{n-1}x$ employed above [26]. For a minimally coupled complex scalar field one has

$$T_{yy} = \frac{1}{2}\left[\partial_y\phi^*\partial_y\phi + a^2 h^{ij}\partial_i\phi^*\partial_j\phi\right] \ . \tag{3.15}$$

Thus, with an integration by parts (certainly allowed for $\phi \in D(A)$) the energy can be written as

$$\begin{aligned} E[\phi] &= \int_\Sigma \sigma d^{n-1}x \ (\partial_y\phi^*\partial_y\phi + \phi^* A\phi) \\ &= (\partial_y\phi, \partial_y\phi) + (\phi, A\phi) \ . \end{aligned} \tag{3.16}$$

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For a comparison of the two definitions (3.14) and (3.16) of the energy and the role of boundary terms, see e.g. the discussion in [27] and the comment in section 3.5 below. Adopting the expression (3.16) as the definition of the energy suggests introducing a Sobolev structure on the space of scalar fields using the quadratic form

$$Q_A(\phi) = (\phi, A\phi) \tag{3.17}$$

associated to A, via [25, 26]

$$||\phi||^2_{H^1} = (\phi, \phi) + Q_A(\phi) \ , \tag{3.18}$$

thus enforcing the condition that the energy be finite. For present purposes we simply note that at least for the Friedrichs extension A_F of A, based on the closure of the quadratic form $Q_A(\phi)$ with respect to the L^2 norm, the resulting potential energy $Q_{A_F}(\phi)$ is finite (and positive) by definition without having to invoke Sobolev spaces (see also the discussion in [28, 29]).[3] We will use specifically this extension in the discussion of section 3.5 below.

3.2 Essential self-adjointness and the Horowitz-Marolf criterion

The spatial part A of the wave operator is real and symmetric (with respect to an appropriate scalar product on a C_0^∞ domain of A), and as such has self-adjoint extensions, each leading to a well defined (and reasonable [28]) time-evolution. If the self-adjoint extension is not unique, however, i.e. if the operator is not essentially self-adjoint, then also the corresponding time-evolution is not uniquely determined. Thus the Horowitz-Marolf criterion [18] (unique time-evolution without having to impose boundary conditions at the singularity) amounts to the condition that the operator A be essentially self-adjoint.

To test for essential self-adjointness [30], one can e.g. use [18] the standard method of Neumann deficiency indices or the Weyl limit point — limit circle criterion (employed in this context in [31]). Roughly speaking, in order for A to be essentially self-adjoint the (effective) potential $V_{\text{eff},\ell}$ appearing in the operator $\tilde{A}_\ell$ has to be sufficiently repulsive near $x = 0$ to prevent the waves $\tilde{\phi}$ from leaking into the singularity.

Concretely, in the present case, where we only have control over the operator A near the singularity at $x = 0$, the criteria for the operator $\tilde{A}_\ell$ to be essentially self adjoint on $C_0^\infty(\mathbb{R}_+)$ at $x = 0$ boil down to the following elementary conditions on the effective potential $W \equiv V_{\text{eff},\ell}$ [30]:

- If
$$W(x) \geq \frac{3}{4} x^{-2} \tag{3.19}$$
near zero, then $-\partial_x^2 + W(x)$ is essentially self-adjoint at $x = 0$.

- If for some $\epsilon > 0$
$$W(x) \leq \left(\frac{3}{4} - \epsilon\right) x^{-2} \tag{3.20}$$

[3] Working with such a Sobolev space structure is certainly possible but also complicates the determination of self-adjoint extensions of A, since e.g. studying the closure of A now involves studying the sixth order operator A^3, arising from the term $||A\phi||^2_{H^1} = (A\phi, A\phi) + (A\phi, A^2\phi)$ in the operator norm.

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(in particular also if $W(x)$ is decreasing) near $x=0$, then $-\partial_x^2+W(x)$ is not essentially self-adjoint at $x=0$.

The significance of the factor 3/4 can be appreciated by looking at the critical (and relevant for us) case of an inverse square potential

$$W(x)=s(s-1)x^{-2} \quad . \tag{3.21}$$

In this case the leading behaviour of the two linearly independent solutions of the equation

$$\left(-\partial_x^2+W(x)\right)\tilde{\phi}_\lambda(x)=\lambda\tilde{\phi}_\lambda(x) \tag{3.22}$$

near $x=0$ is given by the two linearly independent solutions of the equation

$$\left(-\partial_x^2+W(x)\right)\tilde{\phi}_0(x)=0 \quad , \tag{3.23}$$

namely

$$\tilde{\phi}_0\sim x^s \qquad \text{or} \qquad \tilde{\phi}_0\sim x^{1-s} \tag{3.24}$$

Thus both solutions are square integrable near $x=0$ when $2s>-1$ and $2(1-s)>-1$, or

$$-\frac{1}{2}<s<\frac{3}{2} \qquad \Leftrightarrow \qquad s(s-1)<\frac{3}{4} \quad , \tag{3.25}$$

and in this range of $c=s(s-1)$ the potential is limit circle and the self-adjoint extension is not unique. Conversely, it follows that for $c\geq 3/4$ the solutions of equation (3.22) for $\lambda=\pm i$ (which are necessarily complex linear combinations of the two linearly independent real solutions) are not square-integrable near $x=0$. Thus the deficiency indices are zero and the operator is essentially self-adjoint for $c\geq 3/4$.

Even when there are two normalisable solutions, all is not lost however, as it may be indicative of the possibility (or even necessity) to continue the fields and/or the metric through the singularity [22]. Evidently, such an analytic continuation requires some thought (to say the least) in the case of Szekeres-Iyer metrics with generic (non-rational) values of p and q.

3.3 The Horowitz-Marolf criterion for power-law singularities

In the case at hand, timelike singularities of power-law type, the effective potential is given by (2.17) with $\eta=-1$ and $s=qd/4$. We had already seen in section 2.5 that the mass term is never dominant at $x=0$ and we can therefore also set $m^2=0$. Thus the operator of interest is

$$\begin{aligned}\tilde{A}_\ell &= -\partial_x^2+V_{\text{eff},\ell}(x)\\ V_{\text{eff},\ell}(x) &= s(s-1)x^{-2}+\ell_d^2x^{p-q} \quad ,\end{aligned} \tag{3.26}$$

It is now straightforward to determine for which values of (p,q) the classical singularities at $x=0$ become regular or remain singular when probed by scalar waves. First of all, we will show that we can reduce the analysis to the case $\ell=0$:

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- For $q < p+2$, the first term in the potential is dominant and independent of ℓ. Thus A is essentially self-adjoint iff $\tilde{A}_{\ell=0}$ is essentially self-adjoint. As we know from (3.19), this condition is satisfied iff $s(s-1) \geq 3/4$.

- For $q > p+2$, the operators $\tilde{A}_\ell$ for $\ell \neq 0$ are essentially self-adjoint by the criterion (3.19). Thus A is essentially self-adjoint iff $\tilde{A}_{\ell=0}$ is.

- In the borderline case $q = p+2$, for $\ell \neq 0$ we have

$$\ell \neq 0 \quad \Rightarrow \quad s(s-1) + \ell_d^2 \geq 3/4 \tag{3.27}$$

  (with equality only for $s = 1/2$ and $\ell = d = 1$). Even in this case, therefore, all the $\tilde{A}_\ell$ with $\ell \neq 0$ are essentially self-adjoint and only $\tilde{A}_{\ell=0}$ needs to be examined.

We can thus conclude that the operator A is essentially self-adjoint iff $s(s-1) \geq 3/4$ and that, in view of (3.25), it fails to be essentially self-adjoint for

$$A \text{ not e.s.a.} \quad \Leftrightarrow \quad -\frac{1}{2} < s < \frac{3}{2} \quad \Leftrightarrow \quad -\frac{2}{d} < q < \frac{6}{d} \ . \tag{3.28}$$

3.4 The significance of the (strict) dominant energy condition

While this has been rather straightforward, one of the virtues of the present approach, based on using a class of metrics appropriate for a generic singularity of power-law type, is that it allows us to draw a general conclusion regarding the relation between the Horowitz-Marolf criterion and properties of the matter (stress-energy) content of the space-time near the singularity.

Indeed, as we will now show, whenever the matter content of the near-singularity space-time is sufficiently "positive" (in the sense of the strict DEC, as it turns out), the space-time remains singular according to the Horowitz-Marolf criterion, i.e. when probed with scalar waves.

We can deduce from (2.29) that metrics with timelike power-law singularities satisfying the strict DEC lie in a bounded region inside the strip $0 < q < 2/d$ [1]. Indeed, for $q < p+2$ only the second terms in (2.29) are relevant, and one finds

$$\rho - P_r = \frac{1}{4} dq(2-dq)\, x^{-(p+2)} \ . \tag{3.29}$$

Thus one has

$$\rho - P_r > 0 \quad \Leftrightarrow \quad 0 < q < \frac{2}{d} \ . \tag{3.30}$$

In particular, therefore, it follows from (3.28) that for such metrics the operator A is not essentially self-adjoint and we can draw the general conclusion that

> metrics with timelike singularities of power-law type satisfying the strict Dominant Energy Condition remain singular when probed with scalar waves.

Even though metrics with $q = 2/d$, say, like negative mass Schwarzschild, still satisfy the bound (3.28), thus remain singular while obeying an extremal equation of state, we cannot strengthen the above statement to include general metrics with extremal equations of state. This can be seen e.g. from examples in [18] and is due to the fact that extremal metrics can also be found elsewhere in the (p, q)-plane, in particular in the region $q > p+2$, while violating the bound (3.28).

3.5 The Friedrichs extension and "hospitable" singularities

In the previous section we have discussed self-adjoint extensions of (the spatial part A of) the Klein-Gordon operator. We have not discussed, however, what these self-adjoint extensions imply about the behaviour of the allowed scalar fields ϕ (those in the domain of the self-adjoint extension of A) near the singularity at $x = 0$.

It is certainly possible that self-adjointness can be achieved by allowing only scalar fields that vanish at the singularity. In some sense, then, the singularity remains excluded from the space-time and is not probed directly by the scalar field ϕ. We will see that this is indeed what happens in (in a precise sense) one half of the cases in which there is a unique self-adjoint extension.

However, it is a priori also possible (and perhaps more interesting) to have a well-defined time-evolution (which we take to mean "defined by some self-adjoint extension" [28]) with scalar fields that are permitted to be non-zero at the singularity. In that case, the singularity would be probed more directly by the scalar field, and one might then perhaps like to define a classical singularity to be "hospitable" (for a scalar field), if there is a self-adjoint extension which allows the scalar fields to take non-zero values at the locus of the singularity. We will see that this possibility is indeed realised as well, not only for the other half of the essentially self-adjoint cases, but also for e.g. the Friedrichs extension A_F of the operator A in a certain range of parameters for which A is not essentially self-adjoint.

To address these issues, we need to determine the domain of definition of the relevant self-adjoint extension of $\tilde{A}_0 = -\partial_x^2 + cx^{-2}$ for $c = s(s-1) \in [-1/4, \infty)$. For $\tilde{A}_0$ essentially self-adjoint, i.e. $c \geq 3/4$, this can be done by explicitly determining the domain of the closure $\bar{A}_0$ of the operator $\tilde{A}_0$. While we have done this (see also [32]), alternatively, for all $c \geq -1/4$, one can determine the domain of the Friedrichs extension $\tilde{A}_F$ of $\tilde{A}_0$, constructed from the closure of the associated quadratic form. For $c \geq 3/4$, such that $\tilde{A}_0$ is essentially self-adjoint, its unique self-adjoint extension of course agrees with the Friedrichs extension. Precisely this problem has been addressed and solved in [33], and instead of reinventing the wheel here we can draw on the results of that reference to analyse the issue at hand.

The main result of [33] of interest to us is their Theorem 6.4. Applied to the operator $\tilde{A}_0$, this theorem[4] states that the domain of the Friedrichs extension $\tilde{A}_F$ of $\tilde{A}_0$ is

$$\begin{aligned} D(\tilde{A}_F) = \{f \in L^2(0,\infty) : \ & f(0) = 0, f \in A(0,\infty), \partial_x f \in L^2(0,\infty), \\ & x^{-1} f \in L^2(0,\infty), (-\partial_x^2 + cx^{-2}) f \in L^2(0,\infty)\} \end{aligned} \tag{3.31}$$

[4] Actually, in [33] a more general operator, including in particular a non-zero harmonic oscillator term Bx^2, was studied. However, this term serves only to regularise the wave functions at infinity. Since we are concerned with the behaviour at $x = 0$, this term is of no consequence for the present considerations.

where $A(0,\infty)$ denotes the space of absolutely continuous functions. In [33], this result was established for $c > 0$. It is actually true, as it stands, down to (and including) $c = -1/4$. In [32] it was shown that the selfadjoint extensions for $-1/4 < c < 3/4$, i.e. $1/2 < s < 3/2$, are characterised by functions of the form $\tilde{\phi}_0(x) = C_1 x^s + C_2 x^{1-s} + f(x)$, where $f(x)$ is contained in $D(\tilde{A}_F)$ and in addition bounded by $C_B x^{3/2}$ near zero, here C_1, C_2, C_B are constants. By comparison with $D(\tilde{A}_F)$ it remains to check that the Friedrichs extension corresponds to $C_2 = 0$. However as remarked at the end of section 3.1 the Friedrichs extension is given by the closure of the quadratic form $Q_A(\phi) = (\phi, A\phi) = (\phi, C^\dagger C\phi) = (C\phi, C\phi)$ with $C = -\partial_x + s/x$. Here we used equation (2.20) and performed an integration by parts, which causes no boundary problems for $\phi \in C_0(\mathbb{R}_+)$. It is now easy to see that the $\tilde{\phi}_0$ for nonvanishing C_2 are excluded as $(C\tilde{\phi}_0, C\tilde{\phi}_0)$ diverges. We will comment on the special case $c = -1/4$ below.

We will now extract from this result some restrictions on the behaviour of f near $x = 0$ (assuming that we can model the leading behaviour of f as $x \to 0$ by some power of x):

1. From the condition $x^{-1} f \in L^2$ we learn that $f(x) \sim x^{\frac{1}{2}+\epsilon}$ for some $\epsilon > 0$. Then the conditions $f(0) = 0$ and $\partial_x f \in L^2$ are also satisfied.

2. The remaining condition $(-\partial_x^2 + cx^{-2}) f \in L^2$ can be satisfied in one of two ways. Either both terms separately are in L^2 or f lies in the kernel of the operator (as $x \to 0$). In the former case, we find the condition $f(x) \sim x^{\frac{3}{2}+\epsilon}$ with $\epsilon > 0$. In the latter case, since the two functions in the kernel are x^s and x^{1-s}, with (as usual) $c = s(s-1)$, we now need to distinguish several cases:

   (a) $c > 3/4$: this means that $s > 3/2$ or $s < -1/2$. The solution x^s with $s > 3/2$, i.e. $f(x) \sim x^{\frac{3}{2}+\epsilon}$, yields nothing new. The solution x^{1-s} with $s > 3/2$ (or, equivalently, the solution x^s with $s < -1/2$) is ruled out by condition 1.

   (b) $c = 3/4$: this means that $s = 3/2$ or $s = -1/2$. In this case, we can allow $x^{3/2}$ and thus relax the domain to include functions $f(x) \sim x^{\frac{3}{2}+\epsilon}$, now with $\epsilon \geq 0$.

   (c) $-1/4 < c < 3/4$: thus $-1/2 < s < 3/2$ and $s \neq 1/2$. Thus the solution x^s is adjoined to the functions $\{x^{\frac{3}{2}+\epsilon}\}$ for $s > 1/2$, and the solution x^{1-s} for $s < 1/2$.

It remains to discuss the special value $c = -1/4$ or $s = 1/2$ which is not covered by the formulation of the domain in (3.31). This is the minimal allowed value of interest to us ($c = s(s-1)$ with s real), and also the minimal value for which the operator remains positive (and thus has a Friedrichs extension). In this case, the two solutions are $x^s = x^{\frac{1}{2}}$ and $x^{\frac{1}{2}} \log x$, and we checked that, as expected, the domain of the Friedrichs extension includes $x^{1/2}$. This can also be deduced e.g. from [34], which moreover illustrates nicely some of the weirdness of non-Friedrichs extensions.

The above discussion shows that the two definitions (3.14) and (3.16) of the energy, a priori differing by boundary terms due to the integration by parts, agree for the Friedrichs extension for $c > -1/4$ and differ only by a finite term for $c = -1/4$. The issue of boundary terms for more general domains is discussed in [27].

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Returning to the original question of determining the behaviour of the allowed scalar fields in the domain of the self-adjoint extension of the spatial part A of the Klein-Gordon operator, we need to now undo the transformation $\phi \to \tilde{\phi}$ from the initial scalar fields ϕ to the half-densities $\tilde{\phi}$ that we performed in section 2.1 to put A into the form of a standard Schrödinger operator.

This transformation back from $\tilde{\phi}$ to ϕ is accomplished by multiplication by x^{-s}. Now the upshot of the above discussion is that the lowest power of x appearing in the domain of $\tilde{A}_F$ is

$$\tilde{\phi}_{\min} \sim \begin{cases} x^{\frac{3}{2}+\epsilon} & \text{for } \; s > 3/2 \;\text{ or }\; s < -1/2 \\ x^s & \text{for } \; 1/2 \leq s \leq 3/2 \\ x^{1-s} & \text{for } \; -1/2 \leq s \leq 1/2. \end{cases} \tag{3.32}$$

Evidently these functions are, in particular, positive powers of x. Thus they, and therefore all functions in the domain, tend to zero for $x \to 0$, consistent with the condition $f(0) = 0$ in (3.31). However this is not necessarily true for the transformed functions, for which one has ($\delta = \delta(s) > 0$ is a positive real number depending on s)

$$\phi_{\min} = x^{-s}\tilde{\phi}_{\min} \sim \begin{cases} x^{\frac{3}{2}+\epsilon-s} = x^{-\delta} & \text{for } \; s > 3/2 \\ x^0 = 1 & \text{for } \; 1/2 \leq s \leq 3/2 \\ x^{1-2s} = x^{\delta} & \text{for } \; -1/2 \leq s < 1/2 \\ x^{\frac{3}{2}+\epsilon-s} = x^{2+\delta} & \text{for } \; s < -1/2 \end{cases} \tag{3.33}$$

The final result is the simple statement that a ϕ in the domain of the Friedrichs extension A_F of A necessarily goes to zero for $s < 1/2$, ϕ can be non-zero (but remains bounded) for $1/2 < s \leq 3/2$, and can become increasingly singular for large $s > 3/2$.

Note that this statement is not invariant under $s \to 1 - s$. Indeed, while the operator $-\partial_x^2 + s(s-1)x^{-2}$ has this invariance, and therefore also statements about its essential self-adjointness are symmetric under $s \to 1 - s$ (as we have seen), the unitary transformation between ϕ and $\tilde{\phi}$ depends linearly on s and thus leads to a behaviour of the original scalar fields ϕ that does not have this symmetry.

Once again we find a pleasing relation with the DEC, since the watershed happens exactly at $s = 1/2 \Leftrightarrow q = 2/d$ which, as we have seen, corresponds to $\rho = P_r$. Timelike singularities satisfying the strict DEC have $0 < q < 2/d$ (3.30), thus $0 < s < 1/2$. Moreover, metrics with $s \leq -1/2$ have a unique self-adjoint extension ($c \geq 3/4$), thus are regular in the Horowitz-Marolf sense, but are not "hospitable" in the sense described above, while those with $s \geq 3/2$ are.

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